THE DETERMINATION OF A PERMEABILITY FUNCTION FROM CORE MEASUREMENTS AND PRESSURE DATA

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ABSTRACT

The regularized output least squares estimation technique is considered as a procedure to determine the permeability of an aquifer from core measurements of permeability and pressure data. For the numerical discretisation we employ the Rayleigh-Ritz variational method and several numerical examples are discussed.

NOMENCLATURE

G	Matrix
H^1	The Sobolev space of functions in L^2 with first-order derivative in L^2
K	Permeability
K_i	Permeability measurements
K_i^ϵ	Noisy permeability measurements
L	Least-squares functional
L^2	The space of square integrable functions
M	Number of terms expressing $K(X)$
N	Number of terms expressing $p(X)$
N_0	Number of pressure measurements
N_1	Number of permeability measurements
Q_{ad}	Set of admissible permeability functions
a_i	Unknown coefficients expressing $K(x)$
c_i	Unknown coefficients expressing $p(x)$
f	Hydraulic source
p	Pressure
p_i	Pressure measurements
p_i^ϵ	Noisy pressure measurements
x	Space coordinate
x_i	Locations of the pressure measurements
$\tilde{x}_i$	Locations of the permeability measurements
α_i	Functions expressing $p(x)$
γ, λ	Regularization parameters
ϵ	Amount of noise
μ, ν	Positive constants
ψ_i	Functions expressing $K(x)$
$\underline{\rho}$	Vector
$\lVert \cdot \rVert$	L^2-norm

INTRODUCTION

In this paper we study the determination of the reservoir parameter functions from sparse pointwise measurements supplemented with measurements of a nonlinear function of the parameter. The specific application we have in mind is that of determining a permeability function from core measurements and pressure data, [1]. We use a regularized output least squares procedure in which the reservoir mapping is approximated by linear combinations of polynomial functions as in the Rayleigh-Ritz method. The regularization used is the H^1 seminorm that is related to the potential energy functional of an elastic membrane. This regularization gives sufficient compactness to obtain the existence of a solution to the associated minimization problem while implying minimal additional smoothing. Several numerical examples are investigated.

FORMULATION

We study the determination of a spatially dependent permeability mapping $K(x)$ from measurements of the permeability at various locations along with pressure measurements. Towards this end, let $(0,1)$ represent a one-dimensional reservoir and let $K = K(x)$ be a real-valued function defined on $(0,1)$ denoting a permeability function that we wish to estimate. We suppose that measurements $\{p_i\}_{i=\overline{1,N_0}}$ of the pressure p are available at N_0 locations $\{x_i\}_{i=\overline{1,N_0}}$ along with core measurements $\{K_i\}_{i=\overline{1,N_1}}$ of K at N_1 locations

$\{\tilde{x}_i\}_{i=\overline{1,N_1}}$. We assume that at x_{N_0}, the condition $p(x_{N_0})=0$ holds and that the reservoir boundary is insulated. If $f \in L^2(0,1)$ represents a hydraulic source and if $K \in Q_{ad}$, where

$$Q_{ad} := \{K \in H^1(0,1) \mid \infty > \mu \geq K \geq \nu > 0\} \quad (1)$$

where μ and ν are (given) positive constants, then the pressure p is obtained as the unique weak solution in $H^1(0,1)$ of the problem, [2],

$$-(K(x)p'(x))' = f(x), \quad x \in (0,1) \quad (2)$$
$$p'(0) = p'(1) = 0 \quad (3)$$
$$p(x_{N_0}) = 0. \quad (4)$$

Thus, the mapping $Q_{ad} \ni K \mapsto p(K) \in H^1(0,1)$ is well-defined.

REGULARIZED OUTPUT LEAST SQUARES METHOD

The approach we use to recover K from the data obtained from the measurements

$$p(x_i) := p_i, \quad i = \overline{1, N_0} \quad (5)$$
$$K(\tilde{x}_i) := K_i, \quad i = \overline{1, N_1} \quad (6)$$

is the so-called regularized output least squares (ROLS) method, [3]. Hence, we formulate the following minimization problem:

$$\text{minimize}_{K \in Q_{ad}} L(K) \quad (7)$$

where

$$L(K) := \int_0^1 K'^2(x)dx + \gamma \sum_{i=1}^{N_0} (p(x_i;K) - p_i)^2 + \frac{1}{\lambda} \sum_{i=1}^{N_1} (K(\tilde{x}_i) - K_i)^2 \quad (8)$$

where $\gamma, \lambda > 0$ are quantities to be specified. In the right-hand side of eqn.(8), the first term represents the potential energy and imposes that $K \in H^1(0,1)$, whilst the last term may represent prior estimates on the unknowns. It can be shown, [4], that solutions to problem (7) exist.

In order to approximate the problem (7), we express p and K as linear combinations

$$p(x) = \sum_{i=1}^{N} c_i \alpha_i(x), \quad (9)$$

$$K(x) = \sum_{i=1}^{M} a_i \psi_i(x) \quad (10)$$

where $\{\alpha_i\}_{i=\overline{1,N}}$ and $\{\psi_i\}_{i=\overline{1,M}}$ are sets of N and M linearly independent functions in $H^1(0,1)$, respectively. We can choose $\psi_i(x) = x^{i-1}$ for $i = \overline{1,M}$ and $\alpha_i(x) = (x - x_{N_0})x^{i-1}$ for $i = \overline{1,N}$, with the latter choice ensuring that condition (4) is automatically satisfied.

Applying the Rayleigh-Ritz variational method we reduce the problem (2)-(4) to the system of equations

$$G\underline{c} = \underline{\rho} \quad (11)$$

where, $\underline{c} = (c_i)_{i=\overline{1,N}}$, and

$$G_{ij} = \sum_{k=1}^{M} a_k G_{ij}^{(k)},$$

$$G_{ij}^{(k)} = \int_0^1 \psi_k(x)\alpha_i'(x)\alpha_j'(x)dx, \quad i,j = \overline{1,N}, \ k = \overline{1,M} \quad (12)$$

$$\rho_j = \int_0^1 \alpha_j(x)f(x)dx, \quad j = \overline{1,N}. \quad (13)$$

In eqn.(12)

$$G_{11}^{(k)} = 1,$$
$$G_{1j}^{(k)} = \frac{j}{j+k-1} - x_{N_0}\frac{(j-1)}{j+k-2},$$
$$G_{i1}^{(k)} = \frac{i}{i+k-1} - x_{N_0}\frac{(i-1)}{i+k-2},$$
$$G_{ij}^{(k)} = \frac{ij}{i+j+k-2} - x_{N_0}\frac{(2ij-i-j)}{i+j+k-3} + x_{N_0}^2\frac{(i-1)(j-1)}{i+j+k-4} \quad (14)$$

for $i,j = \overline{2,N}$, $k = \overline{1,M}$. Alternatively, one may use the finite element method as described in [5]. From eqns (11)-(13) we have

$$G\underline{c}(\underline{a}) = \underline{\rho}. \quad (15)$$

where $\underline{a} = (a_k)_{k=\overline{1,M}}$. The functional L then takes the form

$$L(\underline{a}) = \gamma\{C_p - 2\underline{\zeta}^T\underline{c}(\underline{a}) + \underline{c}^T(\underline{a})H\underline{c}(\underline{a})\} + \underline{a}^T G_0 \underline{a} + \frac{1}{\lambda}\{C_K - 2\underline{\kappa}^T\underline{a} + \underline{a}^T H_K \underline{a}\} \quad (16)$$

where

$$C_p = \sum_{k=1}^{N_0} p_k^2, \quad C_K = \sum_{k=1}^{N_1} K_k^2,$$

$$\zeta_i = \sum_{k=1}^{N_0} \alpha_i(x_k)p_k = \sum_{k=1}^{N_0}(x_k - x_{N_0})x_k^{i-1}p_k,$$

$$\kappa_n = \sum_{k=1}^{N_1} \psi_n(\tilde{x}_k) K_k = \sum_{k=1}^{N_1} \tilde{x}_k^{n-1} K_k,$$
$$H_{ij} = \sum_{k=1}^{N_0} \alpha_i(x_k)\alpha_j(x_k)$$
$$= \sum_{k=1}^{N_0} (x_k - x_{N_0})^2 x_k^{i+j-2},$$
$$(H_K)_{ln} = \sum_{k=1}^{N_1} \psi_l(\tilde{x}_k)\psi_n(\tilde{x}_k) = \sum_{k=1}^{N_1} \tilde{x}_k^{l+n-2},$$
$$(G_0)_{ln} = \int_0^1 \psi_l'(x)\psi_n'(x)dx$$
$$= \begin{cases} 0, & \text{if } l+n=3 \\ \frac{(l-1)(n-1)}{l+n-3}, & \text{otherwise} \end{cases} \quad (17)$$

for $i,j = \overline{1,N}$ and $l,n = \overline{1,M}$. We can then minimize $L(\underline{a})$ subject to the constraints

$$\infty > \mu \geq \sum_{k=1}^{M} a_k \psi_k(x) \geq \nu > 0, \quad x \in (0,1) \quad (18)$$

using the NAG routine E04UCF, which is based on a sequential programming method designed to minimize an arbitrary smooth function subject to simple bounds on the variables and linear and nonlinear constraints. In the computation, apart from the expression for $L(\underline{a})$ given by eqn.(16), one also needs to supply its gradient which is given by, [1],

$$\nabla L(\underline{a}) = 2[-\frac{1}{\lambda}\underline{\kappa} - \gamma\underline{\Theta} + (G_0 + \frac{1}{\lambda}H_K)\underline{a}] \quad (19)$$

where

$$\Theta_k = \underline{\phi}^T G^{(k)} \underline{c}(\underline{a}), \quad k = \overline{1,M},$$
$$G\underline{\phi} = H\underline{c}(\underline{a}) - \underline{\zeta}. \quad (20)$$

It can be shown, [1], that if we choose the vector $\underline{\rho}$, associated through eqn.(13) with the forcing term $f(x)$, to be sufficiently small then there is at most one solution $K \in \text{int}Q_{ad}$ of (7), and further the solution $\underline{a}$ is differentiable with respect to the data $\underline{\zeta}$. This differentiability provides a tool with which one can investigate the sensitivity of the interior optimal estimators $\underline{a}$ with respect to perturbations in the data $\underline{\zeta}$, see [5]. From (11) and (19) we can give the optimality conditions satisfied by the (interior) solution of the minimization (7) as follows:

$$G\underline{c}(\underline{a}) = \underline{\rho} \quad (21)$$
$$G\underline{\phi} = H\underline{c}(\underline{a}) - \underline{\zeta} \quad (22)$$
$$\Theta_k = \underline{\phi}^T G^{(k)} \underline{c}(\underline{a}), \quad k = \overline{1,M} \quad (23)$$
$$\left(G_0 + \frac{1}{\lambda}H_K\right)\underline{a} - \frac{1}{\lambda}\underline{\kappa} - \gamma\underline{\Theta} = \underline{0} \quad (24)$$

or, in component form

$$\sum_{j=1}^{N}\sum_{k=1}^{M} a_k G_{ij}^{(k)} c_j = \rho_i, \quad (25)$$
$$\sum_{j=1}^{N}\sum_{k=1}^{M} a_k G_{ij}^{(k)} \phi_j = \sum_{j=1}^{N} H_{ij} c_j - \zeta_i, \quad (26)$$
$$\sum_{l=1}^{M}\left[(G_0)_{kl} + \frac{1}{\lambda}(H_K)_{kl}\right] a_l - \frac{1}{\lambda}\kappa_k - \gamma\sum_{i=1}^{N}\phi_i\sum_{j=1}^{N} G_{ij}^{(k)} c_j = 0 \quad (27)$$

for $i = \overline{1,N}$ and $k = \overline{1,M}$.

According to the discrepancy principle criterion the regularization parameters $\gamma > 0$ and $\lambda > 0$ are chosen such that

$$\sqrt{\sum_{i=1}^{N_0}(p(x_i;K) - p_i^\epsilon)^2 + \sum_{i=1}^{N_1}(K(\tilde{x}_i) - K_i^\epsilon)^2} \approx \sqrt{N_0 + N_1 - 1}\mid\epsilon\mid \quad (28)$$

if instead of the exact measurements (5) and (6) we use the noisy perturbed measurements

$$p_i^\epsilon = p_i + \epsilon, \quad i = \overline{1,N_0}, \quad (29)$$
$$K_i^\epsilon = K_i + \epsilon, \quad i = \overline{1,N_1} \quad (30)$$

where ϵ is a (small) amount of noise.

RESULTS

Let us take $x_{N_0} = 1/2$ and consider the following test example:

$$K(x) = 1 + x, \quad f(x) = 6 - 18x^2,$$
$$p(x) = (2x-1)(2x^2 - 2x - 1)/2. \quad (31)$$

Then from eqn.(13) we obtain

$$\rho_j = -\frac{6[j^3 + j^2 + j + 3]}{j(j+1)(j+2)(j+3)}, \quad j = \overline{1,N}. \quad (32)$$

We take $N = 3$, $M = 2$, $N_0 = N_1 = 1$. Then eqns (9), (10), (12)-(14) and (17) give

$$p(x) = (2x-1)(c_1 + c_2 x + c_2 x^2)/2,$$
$$K(x) = a_1 + a_2 x, \quad G = a_1 G^{(1)} + a_2 G^{(2)},$$

$$G^{(1)} = \begin{pmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 7/12 & 7/12 \\ 1/2 & 7/12 & 19/30 \end{pmatrix},$$
$$G^{(2)} = \begin{pmatrix} 1/2 & 5/12 & 5/12 \\ 5/12 & 11/24 & 59/120 \\ 5/12 & 59/120 & 11/20 \end{pmatrix},$$
$$\rho_1 = -\frac{3}{2}, \quad \rho_2 = -17/20, \quad \rho_3 = -7/10,$$
$$G_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \quad (33)$$

The constraint (18) recasts as the simple bounds on the variables

$$\mu \geq a_1 \geq \nu, \quad \mu - \nu \geq a_2 \geq \nu - \mu \qquad (34)$$

and the linear constaint

$$\mu \geq a_1 + a_2 \geq \nu. \qquad (35)$$

We distinguish three cases, as follows.

Case (a) $N_0 = 1,\ N_1 = 2$

In this case we take $\tilde{x}_1 = 0$ and $\tilde{x}_2 = 1$, such that only boundary measurements (5) of K are employed. Then eqns (17) and (20) give

$$p_1 = 0,\ K_1 = 1 + \epsilon,\ K_2 = 2 + \epsilon,\ C_p = 0,$$
$$C_K = (1+\epsilon)^2 + (2+\epsilon)^2,\ \underline{\zeta} = \underline{0},$$
$$\kappa_1 = 3 + 2\epsilon,\ \kappa_2 = 2 + \epsilon,\ H = 0_{3\times 3},$$
$$\underline{\phi} = \underline{0},\ \underline{\Theta} = \underline{0},\ H_K = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}. \quad (36)$$

Then eqn.(16) gives

$$L(a_1, a_2) = a_2^2 + \frac{1}{\lambda}((1+\epsilon)^2 + (2+\epsilon)^2 - 2(3+2\epsilon)a_1 - 2(2+\epsilon) + 2a_1^2 + a_2^2 + 2a_1 a_2) \qquad (37)$$

whilst eqns (19) and (24) give

$$\nabla L(a_1, a_2) = \frac{2}{\lambda}(2a_1 + a_2 - 3 - 2\epsilon, (1+\lambda)a_2 + a_1 - 2 - \epsilon) = (0, 0). \qquad (38)$$

Solving (38) we obtain

$$a_1 = \epsilon + \frac{3\lambda + 1}{2\lambda + 1}, \quad a_2 = \frac{1}{2\lambda + 1}. \qquad (39)$$

Then $K_\lambda^\epsilon(x) = \epsilon + \frac{x + 3\lambda + 1}{2\lambda + 1}$, and according to the discrepancy principle (28) we choose

$$\lambda = \frac{|\epsilon|}{1 - 2|\epsilon|} > 0 \qquad (40)$$

where we assume that $|\epsilon| < 0.5$, i.e. the amount of noise is less than 50% error in the measurement of K_1 and less that 25% error in the measurement of K_2. This gives $a_1^\epsilon = 1 + \epsilon + |\epsilon|$, $a_2^\epsilon = 1 - 2|\epsilon|$, $K^\epsilon(x) := a_1^\epsilon + a_2^\epsilon x = 1 + \epsilon + |\epsilon| + x(1 - 2|\epsilon|)$. Observe that when $\epsilon = 0$, then we obtain $K^0(x) = 1 + x$, which is the exact solution. This shows that the retrieval of the permeability K is unique. We can also evaluate the L^2-norm $||K^\epsilon - K|| = \frac{2}{\sqrt{3}}|\epsilon|$, which shows that retrieval of the permeability K is stable, if the regularization parameter λ is chosen as in eqn.(40). Finally, to determine the pressure we solve the system of eqns (21), i.e.

$$\begin{aligned} &(18 + 12\epsilon)c_1 + (11 + 2\epsilon)c_2 \\ &\quad + (11 + 2\epsilon)c_3 = -18, \\ &10(11 + 2\epsilon)c_1 + 5(25 + 6\epsilon)c_2 \\ &\quad + (129 + 22\epsilon)c_3 = -102, \\ &10(11 + 2\epsilon)c_1 + (129 + 22\epsilon)c_2 \\ &\quad + 2(71 + 10\epsilon)c_3 = -84 \end{aligned} \qquad (41)$$

where for simplicity we have taken $\epsilon \geq 0$. The solution of the system of eqns (41) is given by

$$\begin{aligned} c_1 &= -(8\epsilon^2 + 86\epsilon + 63)/\Delta, \\ c_2 &= -(16\epsilon^2 + 268\epsilon + 126)/\Delta, \\ c_3 &= (-8\epsilon^2 + 232\epsilon + 126)/\Delta. \end{aligned} \qquad (42)$$

where $\Delta = (3 + 2\epsilon)(4\epsilon^2 + 36\epsilon + 21)$. Observe that when $\epsilon = 0$, we obtain $p^0(x) = (2x - 1)(2x^2 - 2x - 1)/2$, which is the exact solution (31). This shows that the retrieval of the pressure p is unique. We can also evaluate the L^2-norm

$$||p^\epsilon - p||^2 = 4\epsilon^2(68\epsilon^4 + 1160\epsilon^3 + 5761\epsilon^2 + 6547\epsilon + 2189)/(35\Delta^2) \qquad (43)$$

which shows that the error norm (44) tends to zero linearly with ϵ, as $\epsilon \to 0$.
From (39)-(42) we obtain $\lambda = 0.0555$ and

$$a_1 = 1.1,\ a_2 = 0.9,\ c_1 = -0.9520,$$
$$c_2 = -1.9719, \quad c_3 = 1.9456$$
$$||K^\epsilon - K|| = 0.0577, \quad ||p^\epsilon - p|| = 0.0120 \quad (44)$$

for $\epsilon = 0.05$, i.e. 5% noise, $\lambda = 0.125$ and

$$a_1 = 1.2,\ a_2 = 0.8,\ c_1 = -0.9090,$$
$$c_2 = -1.9399, \quad c_3 = 1.8912$$
$$||K^\epsilon - K|| = 0.1154, \quad ||p^\epsilon - p|| = 0.0231 \quad (45)$$

for $\epsilon = 0.1$, i.e. 10% noise, and $\lambda = 0.333$ and

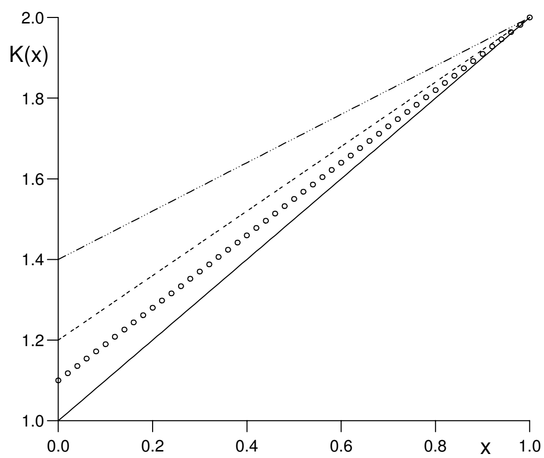
$$a_1 = 1.4,\ a_2 = 0.6,\ c_1 = -0.8350,$$

$$c_2 = -1.8692, \quad c_3 = 1.7846$$
$$||K^\epsilon - K|| = 0.2309, \quad ||p^\epsilon - p|| = 0.0428 \quad (46)$$

for $\epsilon = 0.2$, i.e. 20% noise.

Figure 1 shows the numerical results obtained for various amounts of noise $\epsilon \in \{0.05, 0.1, 0.2\}$. From this figure it can be seen that the numerical solutions for both K and p approach the corresponding analytical solutions (31), as ϵ decreases towards zero.

(i)



(ii)

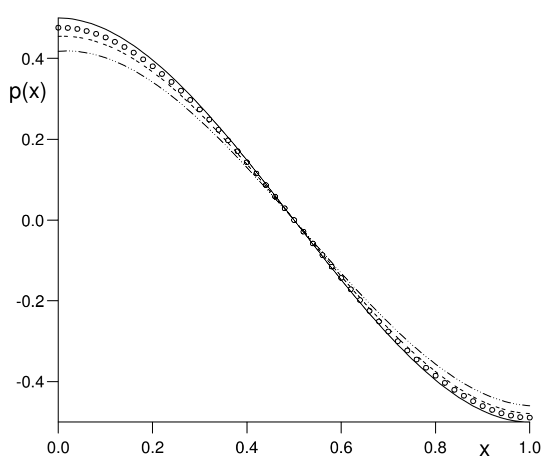


Figure 1: The numerical results for (i) $K(x)$ and (ii) $p(x)$, when $\epsilon = 0.05$ ($\circ\circ\circ$), $\epsilon = 0.1$ ($- - -$) and $\epsilon = 0.2$ ($-...-$), in comparison with the analytical solutions (——).

We note that we could have also proceeded in a simpler way by solving directly (6) to yield $a_1 = 1 + \epsilon$, $a_2 = 1$, and from (21)

$$\begin{aligned} c_1 &= -(54\epsilon^2 + 169\epsilon + 126)/(2\Delta_2), \\ c_2 &= -(66\epsilon^2 + 193\epsilon + 126)/\Delta_2, \\ c_3 &= 2(30\epsilon^2 + 92\epsilon + 63)/\Delta_2, \\ ||K - K^\epsilon|| = \epsilon, \quad ||p - p^\epsilon||^2 &= \epsilon^2(27200\epsilon^4 \\ &+ 163200\epsilon^3 + 360572\epsilon^2 + 347316\epsilon \\ &+ 123227)/(560\Delta_2) \end{aligned}$$

where $\Delta_2 = (3 + 2\epsilon)(10\epsilon^2 + 30\epsilon + 21)$.

Case (b) $N_0 = 3,\ N_1 = 0$

In this case we take $x_1 = 0$ and $x_2 = 1$, such that only boundary measurements of p are employed. Then eqns (17) give

$$\begin{aligned} p_1 = 1/2 + \epsilon,\ p_2 = -1/2 + \epsilon,\ p_3 = 0, \\ C_p = 1/2 + 2\epsilon^2,\ C_K = 0,\ \underline{\kappa} = \underline{0}, \\ \zeta_1 = -1/2,\ \zeta_2 = \zeta_3 = -(1/2 + \epsilon)/2, \\ H_K = 0_{2\times 2},\ H = \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 \end{pmatrix}. \end{aligned} \quad (47)$$

Then eqns (25)-(27) give rise to the following system of 8 nonlinear equations

$$\begin{aligned} 0 &= (60c_1 + 30c_2 + 30c_3)\phi_1 \\ &+ (30c_1 + 35c_2 + 35c_3)\,\phi_2 \\ &+ (30c_1 + 35c_2 + 38c_3)\,\phi_3, \\ \frac{120a_2}{\gamma} &= (60c_1 + 50c_2 + 50c_3)\,\phi_1 \\ &+ (50c_1 + 55c_2 + 59c_3)\,\phi_2 \\ &+ (50c_1 + 59c_2 + 66c_3)\,\phi_3, \quad (48) \end{aligned}$$

$$\begin{aligned} 3(2(c_1 + 1) + c_2 + c_3) &= (12a_1 + 6a_2)\phi_1 \\ &+ (6a_1 + 5a_2)\,\phi_2 + (6a_1 + 5a_2)\,\phi_3, \\ 30(c_1 + c_2 + c_3 + 1 - 2\epsilon) &= \\ (60a_1 + 50a_2)\phi_1 &+ (70a_1 + 55a_2)\phi_2 \\ &+ (70a_1 + 59a_2)\phi_3, \\ 30(c_1 + c_2 + c_3 + 1 - 2\epsilon) &= \\ (60a_1 + 50a_2)\phi_1 &+ (70a_1 + 59a_2)\phi_2 \\ &+ (76a_1 + 66a_2)\phi_3, \quad (49) \end{aligned}$$

$$\begin{aligned} -18 &= (12a_1 + 6a_2)c_1 \\ &+ (6a_1 + 5a_2)\,c_2 + (6a_1 + 5a_2)\,c_3, \\ -102 &= (60a_1 + 50a_2)c_1 + (70a_1 + 55a_2)c_2 \\ &+ (70a_1 + 59a_2)c_3, \\ -84 &= (60a_1 + 50a_2)c_1 + (70a_1 + 59a_2)c_2 \\ &+ (76a_1 + 66a_2)c_3. \quad (50) \end{aligned}$$

with 8 unknowns a_1, a_2, c_1, c_2, c_3, ϕ_1, ϕ_2 and ϕ_3. Solving the system of eqns (50) we obtain

$$\begin{aligned} c_1 &= -(11a_2^2+61a_1a_2+54a_1^2)/(2\Delta_1), \\ c_2 &= (a_2^2-61a_1a_2-66a_1^2)/\Delta_1, \\ c_3 &= 2(a_1^2+32a_1a_2+30a_1^2)/\Delta_1 \end{aligned} \quad (51)$$

where $\Delta_1=(2a_1+a_2)(a_2^2+10a_1a_2+10a_1^2)$.

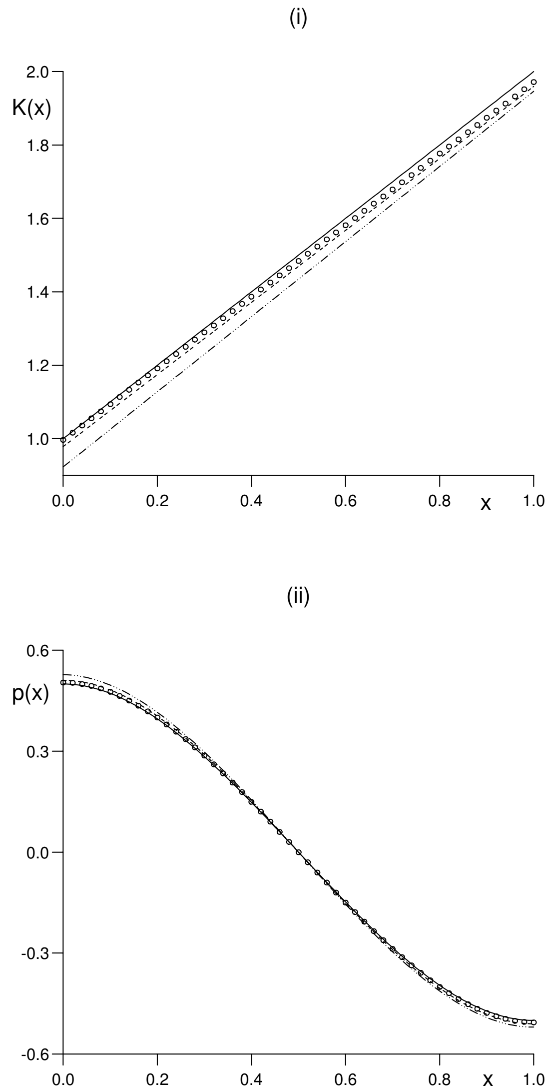


Figure 2: The numerical results for (i) $K(x)$ and (ii) $p(x)$, when $\epsilon=0.05$ ($\circ\circ\circ$), $\epsilon=0.1$ ($---$) and $\epsilon=0.2$ ($-\ldots-$), in comparison with the analytical solutions (—–).

The nonlinear system of eqns (48) and (49) is solved using MAPLE and only the solution which ensures that a_1 and a_2 satisfy the bounds (34) and (35) with $\mu=10^{10}$ and $\nu=10^{-10}$, is accepted. According to the discrepancy principle (28) we have obtained $\gamma=180$ for $\epsilon=0.05$, i.e. 5% noise, and

$$\begin{aligned} &a_1=0.9968,\ a_2=0.9747,\ c_1=-1.0086, \\ &c_2=-2.0258, \quad c_3=2.0226 \\ &\|K^\epsilon-K\|=0.0173, \quad \|p^\epsilon-p\|=0.0036 \end{aligned} \quad (52)$$

$\gamma=90$ for $\epsilon=0.1$, i.e. 10% noise, and

$$\begin{aligned} &a_1=0.9786,\ a_2=0.9811,\ c_1=-1.0211, \\ &c_2=-2.0413, \quad c_3=2.0417 \\ &\|K^\epsilon-K\|=0.0312, \quad \|p^\epsilon-p\|=0.0073 \end{aligned} \quad (53)$$

and $\gamma=45$ for $\epsilon=0.2$, i.e. 20% noise, and

$$\begin{aligned} &a_1=0.9232,\ a_2=1.0229,\ c_1=-1.0552, \\ &c_2=-2.0677, \quad c_3=2.0834 \\ &\|K^\epsilon-K\|=0.0655, \quad \|p^\epsilon-p\|=0.0162. \end{aligned} \quad (54)$$

Figure 2 shows the numerical results obtained for various amounts of noise $\epsilon\in\{0.05,0.1,0.2\}$. From Figures 1 and 2 it can be seen that there is better agreement between the numerical solutions and the analytical solutions for both K and p in Case (b) than in Case (a), and this concludes the fact that the pressure measurements (6) offer more accurate information than the permeability measurements (5).

Case (c) $N_0=2,\ N_1=1$

In this case we take $\tilde{x}_1=0$ and $x_1=1$, such that we measure K at $x=0$ and the pressure p at $x=1$. Then eqns (17) give

$$\begin{aligned} &p_1=-1/2+\epsilon,\ p_2=0,\ K_1=1+\epsilon, \\ &C_p=(-1/2+\epsilon)^2,\ C_K=(1+\epsilon)^2, \\ &\zeta_1=\zeta_2=\zeta_3=-(1/2+\epsilon)/2, \\ &\kappa_1=1+\epsilon,\ \kappa_2=0,\ H_K=\begin{pmatrix}1&0\\0&0\end{pmatrix}, \\ &H=\begin{pmatrix}1/4&1/4&1/4\\1/4&1/4&1/4\\1/4&1/4&1/4\end{pmatrix}. \end{aligned} \quad (55)$$

Then eqns (25)-(27) give rise to the system of three eqns (5) and the following system of 5 nonlinear equations

$$\begin{aligned} \frac{60(a_1-1-\epsilon)}{\lambda\gamma} &= (60c_1+30c_2+30c_3)\phi_1 \\ &+(30c_1+35c_2+35c_3)\,\phi_2 \end{aligned}$$

$$+(30c_1+35c_2+38c_3)\,\phi_3,$$
$$\frac{120a_2}{\gamma}=(60c_1+50c_2+50c_3)\,\phi_1$$
$$+(50c_1+55c_2+59c_3)\,\phi_2$$
$$+(50c_1+59c_2+66c_3)\,\phi_3, \quad (56)$$
$$30(c_1+c_2+c_3+1-2\epsilon)=$$
$$(12a_1+6a_2)\phi_1+(6a_1+5a_2)\,\phi_2$$
$$+(6a_1+5a_2)\,\phi_3,$$
$$30(c_1+c_2+c_3+1-2\epsilon)=$$
$$(60a_1+50a_2)\phi_1+(70a_1+55a_2)\phi_2$$
$$+(70a_1+59a_2)\phi_3,$$
$$30(c_1+c_2+c_3+1-2\epsilon)=$$
$$(60a_1+50a_2)\phi_1+(70a_1+59a_2)\phi_2$$
$$+(76a_1+66a_2)\phi_3 \quad (57)$$

with 8 unknowns a_1, a_2, c_1, c_2, c_3, ϕ_1, ϕ_2 and ϕ_3. Imposing $\lambda=1/\gamma$, according to the discrepancy principle (28) we have obtained $\gamma=41$, for $\epsilon=0.05$, and

$$a_1=1.1165,\ a_2=0.9752,\ c_1=-0.9246,$$
$$c_2=-1.8947,\quad c_3=1.8773$$
$$||K^\epsilon-K||=0.1043,\quad ||p^\epsilon-p||=0.0229. \quad (58)$$

Figure 3 shows the numerical results obtained for the Cases (a)-(c) for $\epsilon=0.05$, i.e. 5% noise added in the input data. From this figure it can be seen that the numerical solutions for $K(x)$ and $p(x)$ provide a stable and reasonable accurate approximation to the analytical solutions (31). More interesting it can be seen that Case (b) provides the best information in the inverse problem. In Case (c) we imposed that $\lambda=1/\gamma$ and used the discrepancy principle (28). Other values of the regularization parameters γ and λ can be selected, and, for example, for $\lambda=1/(10\gamma)$, $\gamma=41$ we obtain

$$a_1=1.0573,\ a_2=1.0608,\ c_1=-0.9449,$$
$$c_2=-1.8887,\quad c_3=1.8892$$
$$||K^\epsilon-K||=0.0895,\quad ||p^\epsilon-p||=0.0192 \quad (59)$$

whilst for $\lambda=1/(100\gamma)$, $\gamma=41$ we obtain

$$a_1=1.0507,\ a_2=1.0704,\ c_1=-0.9473,$$
$$c_2=-1.8879,\quad c_3=1.8904$$
$$||K^\epsilon-K||=0.0883,\quad ||p^\epsilon-p||=0.0188. \quad (60)$$

From eqns (58)-(60) it can be seen that by varying γ and λ independently we can obtain even better estimates for the unknowns than those given by (58). Nevertheless, further investigations are necessary in order to select simultaneously both the regularization parameters γ and λ. A possible choice could be the L-surface method which would extend the L-curve method to two variables.

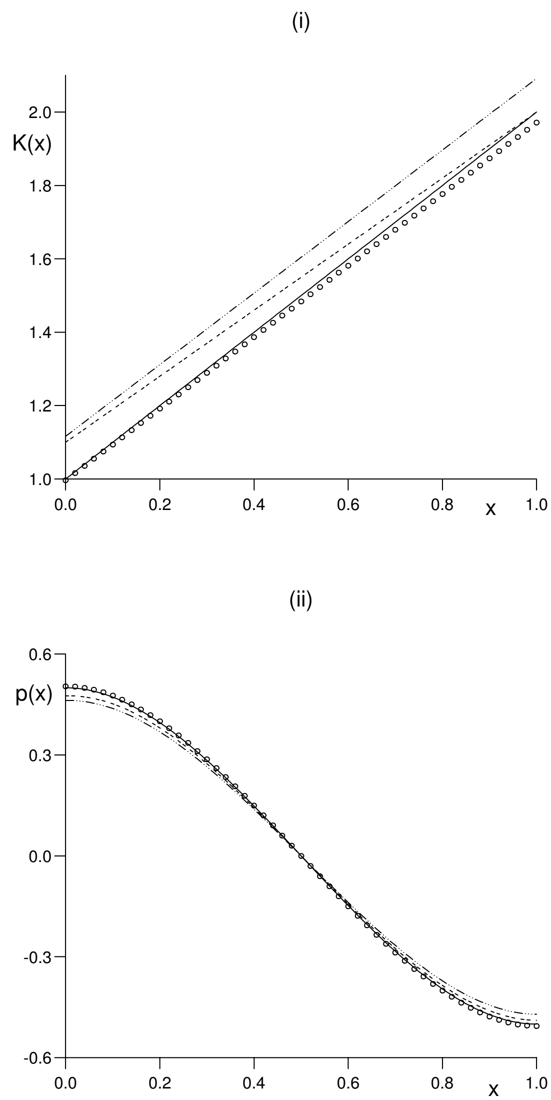


Figure 3: The numerical results for (i) $K(x)$ and (ii) $p(x)$, when $\epsilon=0.05$ for Case (a) (– – –), Case (b) (∘ ∘ ∘), and Case (c) (–...–), in comparison with the analytical solutions (—).

CONCLUSIONS

This study has investigated the application of the regularized output least squares (ROLS) method to the determination of the permeability of a reservoir from core measurements and pressure data. Upon the application of the Rayleigh-Ritz variational method the inverse problem is recast as a constrained minimization problem for which the gradient of the least-squares functional which is minimized can be calculated exactly. Lower dimensional problems can be solved almost analytically using MAPLE, but for higher dimensional parameter identification one needs to produce an appropriate computer programme. It was found that the pressure measurements contain more accurate information about the solution than the permeability measurements themselves, provided that the regularization parameter is properly chosen according to the discrepancy principle. If both core measurements of the permeability and the pressure are used then this involves the selection of two regularization parameters whose proper selection criterion remain yet to be found. Future work will be concerned with the application of the ROLS for the determination of the thermal properties in transient heat conduction, [6].

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REFERENCES

1. L.W. White, Y.-J. Jin and D.J. O'Meara, Estimation of discontinuous elliptic coefficients, *Appl. Math. Comput.*, **81**, 113-138 (1997).

2. O.A. Ladyzhenskaya and N. Ural'tseva, *Linear and Quasilinear Elliptic Equations*, Academic Press, New York, 1968.

3. H.T. Banks and K. Kunisch, *Estimation Techniques for Distributed Parameter Systems*, Birkhauser, Boston, 1989.

5. L.W. White and J. Zhou, Continuity and uniqueness of regularized output least squares optimal estimations, *J. Math. Anal. Appl.*, **196**, 53-83 (1995).

5. L.W. White, Resolution of regularized output least squares estimation procedures, *Appl. Math. Comput.*, **81**, 139-172 (1997).

6. J. Myllymaki and D. Baroudi, *Determination of Thermal Properties. Applications of Regularized Output Least Squares Method*, Espoo 1998, Technical Research Centre of Finland, VTT Publications **351**.